Generalized Coiflets: A New Family of Orthonormal Wavelets

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Abstract

We generalize an existing family of wavelets, coiflets, by replacing the zero-centered vanishing moment condition on scaling functions by a nonzero-centered one in order to obtain a novel class of compactly supported orthonormal wavelets (we call them generalized coiflets). This generalization offers an additional free parameter, i.e., the center of mass of scaling function, which can be tuned to obtain improved characteristics of the resulting wavelet system such as near-symmetry of the scaling functions and wavelets, near-linear phase of the filterbanks, and sampling approximation properties. Therefore, these new wavelets are promising in a broad range of applications in signal processing and numerical analysis.

1. Introduction

In the last decade, the discrete wavelet transform (DWT), which is implemented by a multirate filterbank (FB), has been demonstrated to be a powerful tool for a diversity of digital signal processing applications. Among the numerous wavelets that have been proposed, compactly supported and real-valued wavelets having the orthonormal and perfect reconstruction (PR) properties have been the most widely used. The associated FBs have finite impulse responses (FIR) and real-valued coefficients. In many applications (e.g., image processing), one desirable property for FBs is linear phase, which corresponds to the symmetry or anti-symmetry of the associated wavelets and the symmetry of the associated scaling functions. However, it is well-known that there does not exist any non-trivial, symmetric scaling function in this family [2]; i.e., in order to obtain symmetric scaling functions, at least one of the above properties has to be given up. One possible solution to this dilemma is to construct nearly symmetric scaling functions while maintaining those useful properties, thereby producing FBs with nearly linear phases. The particular class of wavelets known as *coiflets* are near-symmetric. They have the same number of vanishing moments for both the scaling functions (centered at zero) and the wavelets. In addition, coiflets have been shown to be excellent for the sampling approximation of smooth functions.

In this paper, we construct a new family of wavelets (we call them *generalized coiflets*) by replacing the *zero-centered* vanishing moment condition on the scaling functions of coiflets by a *nonzero-centered* one. The merit of such a generalization is that it offers one more free parameter, i.e., the center of mass of the scaling function (denoted by $\bar{t}$), which uniquely characterizes the first several zero-centered moments of scaling functions, and is hence related to the phase response of their FBs at the low frequencies and can be tuned to reduce the phase distortion. In addition, by choosing a proper $\bar{t}$, wavelets that are nearly odd-symmetric are obtained. For a fixed $\bar{t}$, we apply Newton's method to construct the lowpass filters associated with the generalized coiflets iteratively. We show that generalized coiflets are asymptotically symmetric and their filters are asymptotically linear phase as the order tends to infinity. We formulate a general framework for minimizing the phase distortion of the lowpass filters under various criteria. We study the accuracy of generalized coiflets-based sampling approximation of smooth functions by developing the convergence rates for $\boldsymbol{L}^2$-norm of the approximation error. Due to space limitation, we omit all the proofs of our results, which will be given in [4] and [5].

2. Background

We highlight fundamental results from wavelet theory [2] on which this paper is based. Let $h(n)$ be the lowpass filter in a two-channel orthonormal wavelet system. The scaling function $\phi(t)$ is recursively defined by the *dilation equation*

$$\phi(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} h(n)\, \phi(2t - n) \tag{1}$$

and the wavelet $\psi(t)$ is defined as

$$\psi(t) = \sqrt{2} \sum_{n\in\mathbb{Z}} g(n)\, \phi(2t-n)$$

where the lowpass filter h together with a highpass filter g constitute a pair of conjugate quadrature filters (CQFs); i.e.,

$$g(n) = (-1)^n h(n_1+n_2-n) \tag{2}$$

where $[n_1, n_2]$ is the support of the FIR filters h and g. The orthonormal condition is given by

$$\sum_{n\in\mathbb{Z}} h(n)h(n-2k) = \delta_k \tag{3}$$

for $k \in \mathbb{Z}$, where δ_k denotes the Kronecker delta symbol.

3. Definition

Definition: A wavelet is called a *generalized coiflet of order* l (denoted by $\psi_{l,\bar{t}}$) if for some $\bar{t} \in \mathbb{R}$, the wavelet $\psi_{l,\bar{t}}$ and its scaling function (denoted by $\phi_{l,\bar{t}}$) satisfy

$$\int_{\mathbb{R}} (t-\bar{t})^p \phi_{l,\bar{t}}(t)\, dt = \delta_p$$

$$\int_{\mathbb{R}} t^p \psi_{l,\bar{t}}(t)\, dt = 0$$

for $p = 0, 1, \ldots, l-1$.

Note that $\bar{t}$ is the center of mass of the scaling function $\phi_{l,\bar{t}}$. When $\bar{t} = 0$, the generalized coiflets reduce to the original coiflets constructed by Daubechies [3]. The vanishing moment conditions in the above definition are equivalent to

$$\sum_{n\in\mathbb{Z}} n^p h_{l,\bar{t}}(n) = \sqrt{2}\,\bar{t}^p \tag{4}$$

$$\sum_{n\in\mathbb{Z}} (-1)^n n^p h_{l,\bar{t}}(n) = 0 \tag{5}$$

for $p = 0, 1, \ldots, l-1$.

4. Construction

It was shown that a generalized coiflet system can be constructed by solving a set of multivariate nonlinear equations for the filter coefficients [4]. These equations can be derived from (3), (4), and (5). We apply Newton's method to solve them iteratively. Define an $N_l \times 1$ vector

$$\boldsymbol{h}_{l,\bar{t}} \triangleq \left[h_{l,\bar{t}}(-l), h_{l,\bar{t}}(-l+1), \ldots, h_{l,\bar{t}}(N_l-l-1)\right]^T$$

where $N_l = 2\lfloor 3l/2 \rfloor$ and the superscript "T" denotes matrix transpose. Let $\boldsymbol{f}_l : \mathbb{R}^{N_l} \to \mathbb{R}^{N_l}$ be a vector-valued function defined as

$$\boldsymbol{f}_l(\boldsymbol{h}_{l,\bar{t}}) \triangleq \begin{bmatrix} \sum_n h^2_{l,\bar{t}}(n) - 1 \\ \sum_n h_{l,\bar{t}}(n) h_{l,\bar{t}}(n+2) \\ \vdots \\ \sum_n h_{l,\bar{t}}(n) h_{l,\bar{t}}(n+N_l-2) \\ \sum_n n h_{l,\bar{t}}(n) - \sqrt{2}\,\bar{t} \\ \sum_n n^3 h_{l,\bar{t}}(n) - \sqrt{2}\,\bar{t}^3 \\ \vdots \\ \sum_n n^{\lfloor N_l/3 \rfloor -1} h_{l,\bar{t}}(n) - \sqrt{2}\,\bar{t}^{\lfloor N_l/3 \rfloor -1} \\ \sum_n (-1)^n h_{l,\bar{t}}(n) \\ \sum_n (-1)^n n h_{l,\bar{t}}(n) \\ \vdots \\ \sum_n (-1)^n n^{l-1} h_{l,\bar{t}}(n) \end{bmatrix}$$

where all the summations are from $n = -l$ to $n = N_l - l - 1$. Therefore, the equation

$$\boldsymbol{f}_l(\boldsymbol{h}_{l,\bar{t}}) = \boldsymbol{0}_{N_l}$$

gives a set of N_l independent equations in (3), (4), and (5), where $\boldsymbol{0}_{N_l}$ denotes the zero vector of length N_l. The approximate solution to this equation in the kth iteration is denoted by $\boldsymbol{h}^k_{l,\bar{t}}$. With an initialization of $\boldsymbol{h}^0_{l,\bar{t}}$, Newton's iteration becomes

$$\boldsymbol{h}^{k+1}_{l,\bar{t}} = \boldsymbol{h}^k_{l,\bar{t}} - \left(\boldsymbol{f}'_l(\boldsymbol{h}^k_{l,\bar{t}})\right)^{-1} \boldsymbol{f}_l(\boldsymbol{h}^k_{l,\bar{t}})$$

where $\boldsymbol{f}'_l$ denotes the Gateaux-derivative of $\boldsymbol{f}_l$ and the operator $(\cdot)^{-1}$ denotes matrix inversion. The initial choice of $\boldsymbol{h}^0_{l,\bar{t}}$ is not arbitrary because some choices may cause the iteration to diverge. In our design, we choose the original coiflet filter one order lower than the generalized coiflet filter we aim to construct as the starting solution; i.e.,

$$\boldsymbol{h}^0_{l+1,\bar{t}} = \begin{cases} \begin{bmatrix} 0 & (\boldsymbol{h}^0_{l,0})^T & 0 \end{bmatrix}^T & \text{if } l \text{ is even} \\ \begin{bmatrix} 0 & (\boldsymbol{h}^0_{l,0})^T & 0 & 0 & 0 \end{bmatrix}^T & \text{if } l \text{ is odd.} \end{cases}$$

The iteration stops when the difference between $\boldsymbol{h}^{k+1}_{l,\bar{t}}$ and $\boldsymbol{h}^k_{l,\bar{t}}$ is small enough (e.g., its norm is smaller than a given threshold). In our experiments, with such an initialization scheme, the Newton iteration always converges.

5. Near-Symmetry and Near-Linear Phase

We use the phase of the Fourier transform of a scaling function to measure its symmetry. If $\phi_{l,\bar{t}}$ is nearly symmetric, then the phase of $\widehat{\phi}_{l,\bar{t}}$ is close to a linear function of frequency. The following proposition indicates how good this approximation is at low frequencies. Due to the lowpass nature of $\phi_{l,\bar{t}}$, in the frequency domain its energy is mostly distributed at low frequencies. Therefore, $\phi_{l,\bar{t}}$ is nearly symmetric.

Proposition 1: If $|\omega|$ is sufficiently small, then

$$\angle\widehat{\phi}_{l,\bar{t}}(\omega) = -\bar{t}\omega + C_{\phi_{l,\bar{t}}} \cdot \omega^l + O(\omega^{l+1})$$

where the constant $C_{\phi_{l,\bar{t}}}$ only depends on $\phi_{l,\bar{t}}$; the scaling function of a generalized coiflet is asymptotically symmetric, i.e., for each ω,

$$\lim_{l\to\infty} \angle\widehat{\phi}_{l,\bar{t}}(\omega) = -\bar{t}\,\omega.$$

Now we study the phase distortion of the lowpass filters associated with the generalized coiflets. Since the coefficients of these filters are real-valued, we only consider $\omega \in [0, \pi]$. For a lowpass filter, there are two types of symmetry. If for some $\bar{n}$, a filter h satisfies $h(n) = h(2\bar{n} - n)$, then we say that h is *whole-point symmetric* (WPS) about $\bar{n}$ if $\bar{n} \in \mathbb{Z}$, and *half-point symmetric* (HPS) about $\bar{n}$ if $(\bar{n} + \frac{1}{2}) \in \mathbb{Z}$. In both cases, the phase response $\angle H(e^{j\omega}) = -\bar{n}\omega$. If a filter is asymmetric, then its phase distortion can be measured as the deviation of the phase response from a linear function of frequency with some desired slope.

There is a well-known fact [1] regarding the relationship between the symmetric type of a wavelet ψ and its associated lowpass filter h: if h is WPS, then ψ is even-symmetric, and vice versa; if h is HPS, then ψ is odd-symmetric, and vice versa. It has been observed that the lowpass filters associated with the original coiflets are nearly WPS [3]. In fact, for an lth-order original coiflet, its filter $H_{l,0}(e^{j\omega})$ has $(l-1)$ zeros at $\omega = 0$, and hence a flat, near-zero phase in the neighborhood of DC. In the following proposition, we show that in general, $H_{l,\bar{n}}(e^{j\omega})$ is close to linear phase at low frequencies.

Proposition 2: If $|\omega|$ is sufficiently small, then

$$\angle H_{l,\bar{n}}(e^{j\omega}) = -\bar{n}\omega + C_{h_{l,\bar{n}}} \cdot \omega^l + O(\omega^{l+2})$$

where the constant $C_{h_{l,\bar{n}}}$ only depends on $h_{l,\bar{n}}$; the lowpass filter associated with a generalized coiflet possesses asymptotically linear phase, i.e., for each $\omega \in [0, \pi)$,

$$\lim_{l\to\infty} \angle H_{l,\bar{n}}(e^{j\omega}) = -\bar{n}\omega.$$

Though the above proposition is true for all real $\bar{n}$, only integers and half-integers are of interest. Thus, we define $\mathbb{Z}_2 \stackrel{\triangle}{=} \{n : 2n \in \mathbb{Z}\}$. An advantage of introducing half-integer $\bar{n}$ is that filters close to HPS can be constructed, which are more useful than WPS filters in many applications.

6. Minimization of Phase Distortion

From Proposition 2 we know that the filter $h_{l,\bar{n}}$, $\bar{n} \in \mathbb{Z}_2$, has nearly zero phase distortion if $|\omega|$ is small enough. However, the phase distortion at the other frequencies can be much larger. The resulting phase response may not be satisfactory in many applications that require uniformly insignificant phase distortion over a broad frequency band.

The phase response at low frequencies is uniquely characterized by the first several moments of the scaling function, and hence by the parameter $\bar{t}$ in the case of the generalized coiflets. Thus, we attempt to use this parameter to obtain smaller phase distortion. Though, for any $\bar{t} \notin \mathbb{Z}_2$ the property of near-zero phase distortion around DC will be lost, the gain lies in the fact that phase distortion can be largely reduced over a broad frequency band. For a given $\bar{t} \in \mathbb{R}$, we expect that $\angle H_{l,\bar{t}}(e^{j\omega})$ is close to $-\frac{[2\bar{t}]}{2}\omega$.

While adjusting the parameter $\bar{t}$ may also improve the near-symmetry of the scaling function $\phi_{l,\bar{t}}$ and the wavelet $\psi_{l,\bar{t}}$, in this paper we restrict our attention to the phase distortion of the lowpass filter $h_{l,\bar{t}}$.

Since in a typical DWT-based application the input signal is convolved with a wavelet filterbank, the phase response of the output signal is a sum of those of the input signal and the filterbank. Therefore, the phase distortion on the output signal, which is additive and caused by the non-linearity of the phase response of the filterbank, can be viewed as the difference between the desired linear phase response for the filterbank and its actual phase response. We define $\mathcal{D}_p[h]$ to be the measure of phase distortion of a filter h over $[0, \pi]$,

$$\mathcal{D}_p[h] \stackrel{\triangle}{=} \left\| W(\omega) \left(\angle H(e^{j\omega}) + \bar{n}\omega \right) \right\|_p \qquad (6)$$

for some p, $1 \le p \le \infty$, $\bar{n} \in \mathbb{Z}_2$, and some weighting function $W : [0, \pi] \to [0, 1]$. For the generalized coiflets, $\bar{n} = \frac{[2\bar{t}]}{2}$. In fact, the quantity $\mathcal{D}_p[h]$ is the weighted $\boldsymbol{L}^p$-distance between the desired linear phase response $-\bar{n}\omega$ and the actual phase response $\angle H(e^{j\omega})$. From (2) we deduce that for a CQF pair h and g of a finite support $[n_1, n_2]$,

$$\angle G(e^{j\omega}) = \angle H(e^{j(\pi-\omega)}) - (n_1 + n_2)\omega + \pi.$$

Using this relationship, we rewrite $\mathcal{D}_p[h]$ as

$$\mathcal{D}_p[h] = \left\| \widetilde{W}(\omega) \left(\angle G(e^{j\omega}) + \bar{n}'\omega + (\bar{n} - 1)\pi \right) \right\|_p$$

where $\bar{n}' = n_1 + n_2 - \bar{n}$, $\widetilde{W}$ is the mirror function of W about $\frac{\pi}{2}$, i.e., $\widetilde{W}(\omega) = W(\pi - \omega)$ for $\omega \in [0, \pi]$, and $-\bar{n}'\omega - (\bar{n} - 1)\pi$ is the desired generalized linear phase response. Therefore, the quantity $\mathcal{D}_p[h]$ also measures the phase distortion of g with respect to the weighting function $\widetilde{W}$. Thus, such a metric is meaningful in not only the

DWT-based applications but also those based on wavelet packet transforms, where both lowpass and highpass subbands are decomposed iteratively. With this quantitative measure, which is clearly a function of the parameter $\bar{t}$ for the generalized coiflets, we can formulate a class of optimization problems by searching the optimal parameter $\bar{t}^*$ that minimizes $\mathcal{D}_p[h_{L,\bar{t}}]$ for given p and W. Such a general formulation allows the flexibility of choosing a proper parameter p and a proper weighting function W in order to provide an appropriate filterbank for a particular DWT-based application. In the following examples, we choose $p = \infty$, $W(\omega) = 1$ if $\omega \in [0, \frac{\pi}{2})$, and $W(\omega) = 0$ elsewhere. This implies that we attempt to minimize the maximum phase distortion over the lowpass halfband.

In Figure 1, we plot the original coiflet and the optimal generalized coiflet of order 3 as well as their scaling functions, where the subscript "w" stands for WPS. The optimal generalized coiflet appears more symmetric than the original coiflet. The optimal near-HPS filters of the generalized coiflets with odd orders are quite similar to the filters of some biorthogonal spline wavelets, which are, in fact, the dual wavelets with respect to the Haar wavelet, and referred to as ${}_{1,\widetilde{N}}\widetilde{h}$ in Table 6.1 in [1]. In Figure 2, we plot the order-3 biorthogonal spline wavelet dual to the Haar wavelet and the order-3 generalized coiflet having the minimal phase distortion, as well as their scaling functions, where the subscript "h" stands for HPS. The two scaling functions are surprisingly similar to each other; so are the two wavelets.

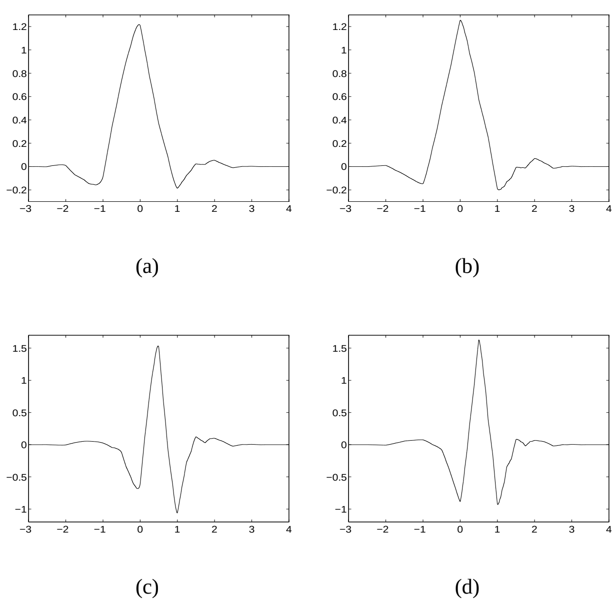


Figure 1. Comparison between the 3rd-order original coiflet and the 3rd-order generalized coiflet having the minimax phase distortion: (a) $\phi_{3,0}(t)$; **(b)** $\phi_{3,\bar{t}^*_{\mathrm{w}}}(t)$; **(c)** $\psi_{3,0}(t)$; **(d)** $\psi_{3,\bar{t}^*_{\mathrm{w}}}(t)$.

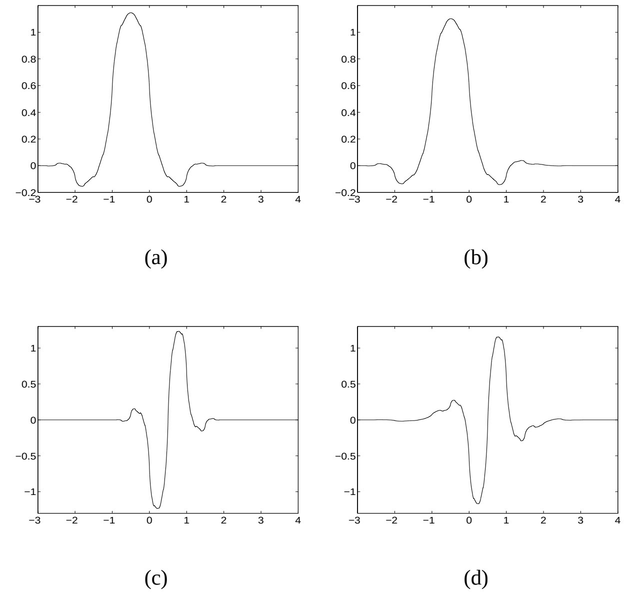


Figure 2. Comparison between the 3rd-order biorthogonal spline wavelet dual to the Haar wavelet and the 3rd-order generalized coiflet having minimax phase distortion: (a) ${}_{1,3}\widetilde{\phi}(t)$; **(b)** $\phi_{3,\bar{t}^*_{\mathrm{h}}}(t)$; **(c)** ${}_{1,3}\widetilde{\psi}(t)$; **(d)** $\psi_{3,\bar{t}^*_{\mathrm{h}}}(t)$.

7. Approximation of Smooth Functions

An important issue in wavelet-based multiresolution approximation theory is to measure the decay of approximation error as resolution increases, given some smoothness conditions on the function being approximated.

Let f be a smooth $\boldsymbol{L}^2$ function in the sense that $f^{(l)}$ is square integrable, and ϕ be an lth-order orthonormal scaling function. Define $\mathcal{P}_i f$ to be the approximation of f at resolution 2^{-i}; i.e., the orthogonal projection on the subspace spanned by $\{\phi^{i,k}\}_k$,

$$(\mathcal{P}_i f)(t) = \sum_{k\in\mathbb{Z}} \langle f, \phi^{i,k}\rangle\, \phi^{i,k}(t) = \sum_{q=i+1}^{+\infty} \sum_{k\in\mathbb{Z}} \langle f, \psi^{q,k}\rangle\, \psi^{q,k}(t).$$

where $\phi^{i,k}(t) = 2^{i/2}\phi(2^i t - k)$, for $i, k \in \mathbb{Z}$, and similar notation applies to ψ.

It can be shown that the $\boldsymbol{L}^2$-norm of the projection error has the asymptotic form

$$\|f - \mathcal{P}_i f\|_2 = C_{\text{proj}} \cdot 2^{-il} \cdot \|f^{(l)}\|_2 + O(2^{-i(l+1)})$$

where the constant C_{proj} is given by

$$C_{\text{proj}} = \frac{1}{l!} \left(\sum_{m \in \mathbb{Z}, m \neq 0} \left| \widehat{\phi}^{(l)}(2m\pi) \right|^2 \right)^{1/2} .$$

In the above discussion, the expansion coefficients $\{\langle f, \phi^{i,k} \rangle\}_k$ are assumed to be available so that the wavelet coefficients $\{\langle f, \psi^{q,k} \rangle : q, k \in \mathbb{Z}, q > i\}$ can be efficiently computed via Mallat's algorithm, on which the multiresolution analysis is based. However, in practice, only the uniform samples of a function rather than its expansion coefficients are often known, because the explicit forms of the function and the scaling function are unknown (this is true for most wavelet bases), and the computation of expansion coefficients usually requires the evaluation of numerical integrals, which are computationally expensive. If a generalized coiflet basis is used, then the function samples approximate the expansion coefficients accurately.

We define a sequence of functions $\{f_{i,l} : \mathbb{R} \to \mathbb{R}, i \in \mathbb{Z}\}$,

$$f_{i,l}(t) = 2^{-\frac{i}{2}} \sum_{k \in \mathbb{Z}} f(2^{-i}(k + \bar{t}))\, \phi^{i,k}_{l,\bar{t}}(t)$$

which can be viewed as successive approximations of f with the scaled and translated scaling functions of an order-l generalized coiflet being used as the interpolants.

Theorem: If $f : \mathbb{R} \to \mathbb{R}$ is $(l+1)$ times differentiable, $f^{(l+1)}$ is bounded, and $f^{(l)} \in \boldsymbol{L}^2(\mathbb{R})$, then the $\boldsymbol{L}^2$-norm of the reconstruction error has the asymptotic form

$$\|f - f_{i,l}\|_2 = C_{\text{appr}} \cdot 2^{-il} \cdot \|f^{(l)}\|_2 + O(2^{-i(l+1)})$$

where the constant C_{appr} is given by

$$C_{\text{appr}} = \frac{1}{l!} \left[\int_0^1 \left(\sum_{k \in \mathbb{Z}} (t - k - \bar{t})^l \phi_{l,\bar{t}}(t-k) \right)^2 dt \right]^{1/2} .$$

In [5], we show that for a generalized coiflet, the asymptotic constant C_{appr} can be expressed as

$$C^2_{\text{appr}} = C^2_{\text{proj}} + C^2_{\text{samp}}$$

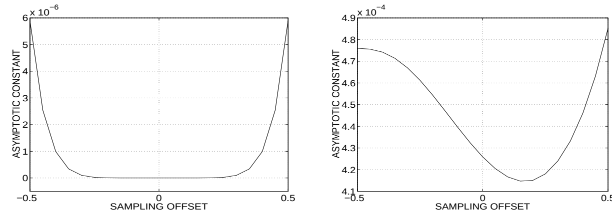
where the constant C_{samp} is given by

$$C_{\text{samp}} = \frac{1}{l!} \left(\int_{\mathbb{R}} t^l \phi_{l,\bar{t}}(t)\, dt - \bar{t}^{\,l} \right) .$$

The squared $\boldsymbol{L}^2$-error for generalized coiflets-based sampling approximation can be represented as

$$\|f - f_{i,l}\|_2^2 = \|f - \mathcal{P}_i f\|_2^2 + \|\mathcal{P}_i f - f_{i,l}\|_2^2.$$

which implies that $\|f - f_{i,l}\|_2^2$, $\|f - \mathcal{P}_i f\|_2^2$, and $\|\mathcal{P}_i f - f_{i,l}\|_2^2$ have the same convergence rate 2^{-2il}. Thus, it is interesting to compare the associated asymptotic constants of



(a) (b)

Figure 3. Asymptotic constants for the sampling approximation errors vs. the sampling offset $\bar{t}$ of the 4th-order generalized coiflet: (a) C^2_{samp}; (b) C^2_{appr}.

the latter two. The asymptotic constant for $\|f - \mathcal{P}_i f\|_2^2$, which is the error due to projection, is apparently C_{proj}. Therefore, C_{samp} is the asymptotic constant for $\|\mathcal{P}_i f - f_{i,l}\|_2^2$, which is the error due to the approximation of the projection coefficients by the function samples.

In [5], we proposed a numerical method to compute the asymptotic constants. Figure 3 illustrates the asymptotic constants for the sampling approximation error versus the sampling offset $\bar{t}$ for the generalized coiflet of orders 4. We find that C_{samp} is much smaller than C_{appr}. This implies that the $\boldsymbol{L}^2$-error due to the approximation of the projection coefficients by the function samples is negligible compared to that due to the projection, because the two types of errors have the same convergence rate.

8. Conclusion

We have presented a study of the generalized coiflets. Since they possess several remarkable properties, they are excellent candidates of compactly supported orthonormal wavelet bases in signal processing and numerical analysis.

References

[1] A. Cohen, I. Daubechies, and J.-C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Commun. Pure Appl. Math.*, 45:485–560, 1992.

[2] I. Daubechies. *Ten Lectures on Wavelets.* SIAM, Philadelphia, PA, 1992.

[3] I. Daubechies. Orthonormal bases of compactly supported wavelets II. variations on a theme. *SIAM J. Math. Anal.*, 24(2):499–519, Mar. 1993.

[4] D. Wei and A. C. Bovik. Generalized coiflets with nonzero-centered vanishing moments. *IEEE Trans. Circuits Syst. II*, Special Issue on Multirate Systems, Filter Banks, Wavelets, and Applications, to appear 1998.

[5] D. Wei and A. C. Bovik. Sampling approximation of smooth functions via generalized coiflets. *IEEE Trans. Signal Processing*, Special Issue on Theory and Applications of Filter Banks and Wavelets, to appear 1998.